Journal of Pure and Applied Mathematics: Advances and Applications
Volume 2, Number 1, 2009, Pages 69-86

A PARTIAL PROOF OF THE ERDŐS-SZEKERES CONJECTURE FOR HEXAGONS

KNUT DEHNHARDT[1], HEIKO HARBORTH[1] and ZSOLT LÁNGI[2]

[1]Diskrete Mathematik
TU Braunschweig
38023 Braunschweig
Germany
e-mail: h.harborth@tu-bs.de

[2]Department of Geometry
Budapest University of Technology
Egry József U. 1., Budapest 1111
Hungary

Abstract

Erdős and Szekeres [5] made the conjecture that, for $n \geq 3$, any set of $2^{n-2}+1$ points in the plane, in general position, contains n points in convex position. A computer-based proof of this conjecture for $n = 6$ appeared in [9] of Peters and Szekeres. The aim of this paper is to give a partial proof of the conjecture for $n = 6$, without the use of computers, for the special case when the convex hull of the point set is a pentagon.

2000 Mathematics Subject Classification: 52A10, 52C35, 52B05.

Keywords and phrases: happy ending problem, Erdős-Szekeres conjecture, hexagon, convex position.

Partially supported by the Alberta Ingenuity Fund.

Received July 25, 2009

1. Introduction

In the early 1930s, Esther Klein asked whether there is an integer N, for every $n \geq 3$, such that any planar set of N points in general position contains n points in convex position. Erdős and Szekeres [5] showed the existence of such an integer, and also that there is a solution satisfying $N \leq \binom{2n-4}{n-2} + 1$. This problem is well-known as the “happy ending problem”.

The task that arose naturally was to find the smallest value $g(n)$ of card S with the mentioned property for each S. In [5], the authors made the following conjecture.

Conjecture 1 (Erdős-Szekeres Conjecture)**.** Let $n \geq 3$. Then the smallest number $g(n)$ such that every planar set of $g(n)$ points in general position contains n points in convex position, is $2^{n-2} + 1$.

In [6], Erdős and Szekeres constructed a planar set of 2^{n-2} points in general position that does not contain n points in convex position. Presently, the best known bounds are

$$2^{n-2} + 1 \leq g(n) \leq \binom{2n-5}{n-2} + 1.$$

The upper bound is due to Tóth and Valtr [10].

Another attempt is to verify the conjecture for small values of n. Note that three points in general position are in convex position. Thus, clearly $g(3) = 3$. The value of $g(4)$ was determined by Esther Klein in the early 1930s.

According to [9], Makai was the first to prove the equality $g(5) = 9$, but he has never published his result. The first published proof appeared in 1970 in [7]. In 1974, Bonnice [2] gave a simple and elegant proof of the same result. In [1], Bisztriczky and Tóth also mention an unpublished proof by Böröczky and Stahl.

The case $n = 6$ seems considerably more complicated. Bonnice [2] makes the following comparison. In a set of nine points, we have $\binom{9}{5} =$

126 possibilities for five points to be in convex position, whereas in a set of seventeen points, we have $\binom{17}{6} = 12376$ possibilities for six points to be in convex position.

For this case, a computer-based proof has been given recently by Peters and Szekeres [9]. In their paper, they used a computer to re-prove the case $n = 5$. They remark that to prove the $n = 5$ case required less than one second "using a 1.5 GHz workstation", whereas, for the case of convex hexagons, "the total computing time . . . was approximately 3,000 GHz hours". For other results related to the Erdős-Szekeres conjecture, the reader is referred to [3] or [8].

In this paper, we examine the $n = 6$ case of the Erdős-Szekeres conjecture without the use of computers. If $S = \{a_i : i = 1, 2, \ldots, k\}$ is a finite point set in $\mathbb{E}^2$, we denote the convex hull of S by $[S]$ or by $[a_1, a_2, \ldots, a_k]$. For $S_1, S_2, \ldots, S_k \subset \mathbb{E}^2$, we set $[S_1, S_2, \ldots, S_k] = [S_1 \cup S_2 \cup \ldots \cup S_k]$. By $V(P)$, we denote the vertex set of the convex polygon P. Our main result is the following theorem.

Theorem 1. *Let $S \subset \mathbb{E}^2$ be a set of seventeen points in general position and $P = [S]$ be a pentagon. Then S contains six points in convex position.*

We note that a different proof of the same statement appeared in the diploma thesis [4] of one of the authors, Dehnhardt, in 1981.

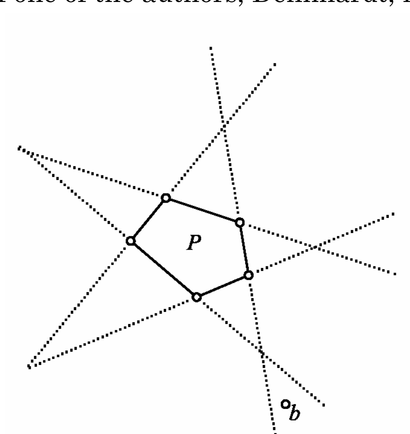

Figure 1. A point b beyond exactly three edges of P.

There are two known sets of sixteen points in general position that do not contain the vertices of a convex hexagon: cf. [6] and pp. 331-332 of [3]. We note that in both examples, the convex hulls of the points are pentagons. We present the proof of Theorem 1 in the next section. We note that, using the same tools, it may be shown that every planar set of twenty five points in general position contains six points in convex position. We also observe that, by Lemma 4, our proof yields that if $S \subset \mathbb{E}^2$ is a set of seventeen points in general position, $P = [S]$ is a triangle or a quadrangle, $Q = [S \setminus V(P)]$ is a pentagon, and $R = [S \setminus (V(P) \cup V(Q))]$ is a triangle, then S contains six points in convex position. Thus, according to the classification of planar point sets introduced by Bonnice in [2], our proof yields the Erdős-Szekeres hexagon conjecture for twenty four classes of point sets out of seventy two. We remark that in [2], it is stated incorrectly that the number of classes is seventy. The correct number (and the list of the classes) can be found, for example, in [4].

In the proof, for two distinct points $a, b \subset \mathbb{E}^2$, $[a, b]$, $L(a, b)$, $L^+(a, b)$, and $L^-(a, b)$ denote, respectively, the closed segment with endpoints a and b, the line containing a and b, the closed ray emanating from a and containing b, and the closed ray emanating from a in $L(a, b)$ that does not contain b. Furthermore, if $s \geq 3$, P is a convex s-gon, and a point $b \in \mathbb{E}^2$ does not lie on any sideline of P and is strictly separated from P by exactly m sidelines of P, we say that b is *beyond exactly m edges of P* (cf. Figure 1). If these sidelines are the lines passing through the edges $E_1, E_2, \ldots, E_m$ of P, we may say that b is *beyond exactly the edges* $E_1, E_2, \ldots, E_m$ of P. For simplicity, a k-gon means a convex k-gon for every $k \geq 3$, and if a set contains six points in convex position, we say that it contains a hexagon.

2. Proof of Theorem 1

We begin the proof with a series of lemmas.

Lemma 1. *Let P and Q be polygons with $Q \subset \operatorname{int} P \subset \mathbb{E}^2$. Let $X \subset V(P)$ be a set of points beyond exactly the same edge of Q. Then $V(Q) \cup X$ is a set of points in convex position* (*cf. Figure* 2).

Lemma 2. *Let* $\{P_i : i = 1, 2, \dots, m\}$ *be a family of* t *triangles,* q *quadrangles, and* p *pentagons such that* $p + q + t = m$ *and* $M = [P_1, P_2, P_3, \dots, P_m]$ *be an* m*-gon* $[x_1, x_2, x_3, \dots, x_m]$. *Suppose that* $[x_i, x_{i+1}]$ *is an edge of* P_i, *and* P_i *and* P_{i+1} *have disjoint interiors for* $i = 1, 2, 3, \dots, m$. *Let* P_0 *be a* k*-gon that contains* M *in its interior and assume that the points of* $W = \bigcup_{i=0}^{m} V(P_i)$ *are in general position. If* $q + 2t < k$, *then* W *contains a hexagon.*

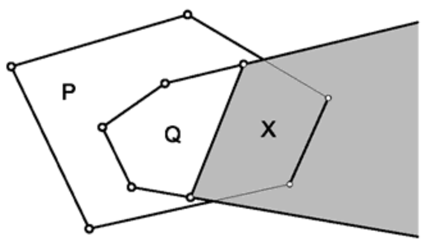

Figure 2

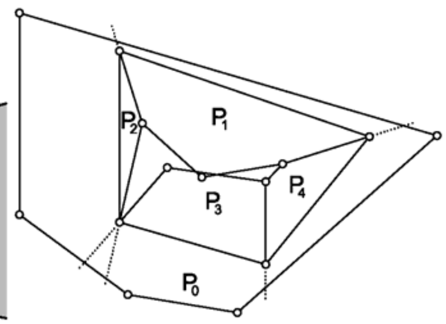

Figure 3

Proof. Let us denote by X_i the set of points that are beyond exactly the edge $[x_i, x_{i+1}]$ of P_i, and observe that every vertex of P_0 is contained in X_i for some value of i. If $\operatorname{card}(X_i \cap V(P_0)) + \operatorname{card}(V(P_i)) \geq 6$ for some P_i, then the assertion follows from Lemma 1 (cf. Figure 3). Thus, we may assume that $\operatorname{card}(X_i \cap V(P_0))$ is at most two, if P_i is a triangle, at most one if P_i is a quadrangle, and zero if P_i is a pentagon, which yields that $k = \operatorname{card}(V(P_0)) \leq 0 \cdot p + 1 \cdot q + 2 \cdot t$, a contradiction. □

We use Lemma 2 often during the proof with $k = 5$. For simplicity, in such cases we use the notation $P_1 * P_2 * \dots * P_m$.

Lemma 3. *Let* $S \subset \mathbb{E}^2$ *be a set of eleven points in general position such that* $P = [S]$ *is a pentagon,* $Q = [S \setminus V(P)]$ *is a triangle, and*

$[S \setminus (V(P) \cup \{q\})]$ *is a quadrilateral for every* $q \in V(Q)$. *Then S contains a hexagon.*

Proof. Note that as card $S = 11$, P is a pentagon, and Q is a triangle, $R = [S \setminus (V(P) \cup V(Q))]$ is a triangle. Let $Q = [q_1, q_2, q_3]$ and $R = [r_1, r_2, r_3]$. Observe that for any $i \neq j$, the straight line $L(r_i, r_j)$ strictly separates the third vertex of R from a unique vertex of Q. We may label our points in a way that q_1, q_2, and q_3 are in counterclockwise cyclic order, and $L(r_i, r_j)$ separates r_k and q_k for any $i \neq j \neq k \neq i$. Let us denote by Q_k the open convex domain bounded by $L^-(q_k, r_i)$ and $L^-(q_k, r_j)$ for every $i \neq j \neq k \neq i$. For every $i \neq j$, let Q_{ij} denote the open convex domain that is bounded by the rays $L^-(q_i, r_j)$, $L^-(q_j, r_i)$, and the segment $[q_i, q_j]$ (cf. Figure 4).

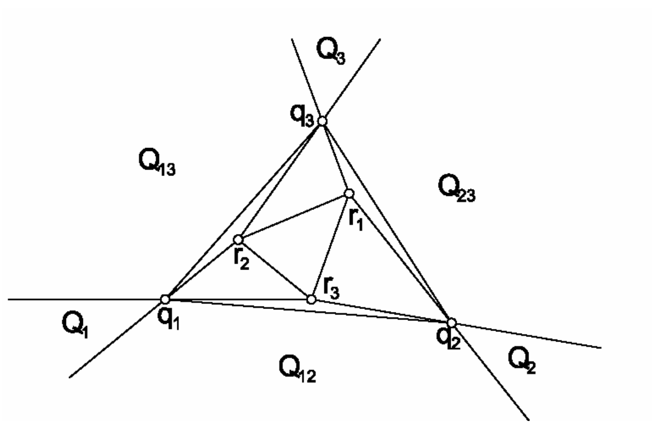

Figure 4

Observe that if Q_{12} contains at least two vertices of P, then these vertices together with q_1, q_2, r_1, and r_2 are vertices of a hexagon. Similarly, if $Q_1 \cup Q_{13} \cup Q_3$ contains at least three vertices of P, or $Q_2 \cup Q_{23} \cup Q_3$ contains at least three vertices of P, then S contains a hexagon. Since P is a pentagon, we may assume that Q_{12} contains one, $Q_1 \cup Q_{13}$ and $Q_2 \cup Q_{23}$ both contain two, and Q_3 contains no vertex of P. By symmetry, we obtain that S contains a hexagon unless $\operatorname{card}(Q_i \cap V(P)) = 0$, and $\operatorname{card}(Q_{ij} \cap V(P)) = 1$ for every $i \neq j$. Since the latter case contradicts the condition that P is a pentagon, S contains a hexagon. □

Lemma 4. *Let $S \subset \mathbb{E}^2$ be a set of thirteen points in general position such that $P = [S]$ is a pentagon and $Q = [S \setminus V(P)]$ is a triangle. Then S contains a hexagon.*

Proof. Let q_1, q_2, and q_3 be the vertices of Q in counterclockwise cyclic order and let $R = S \setminus (V(P) \cup V(Q))$. Observe that card $R = 5$. Using an idea of Klein and Szekeres, we obtain that R contains an empty quadrilateral. In other words, there is a quadrilateral U that satisfies $V(U) \subset R$ and $U \cap R = V(U)$. Let r_1, r_2, r_3, and r_4 be the vertices of U in counterclockwise cyclic order, and let r be the remaining point of R.

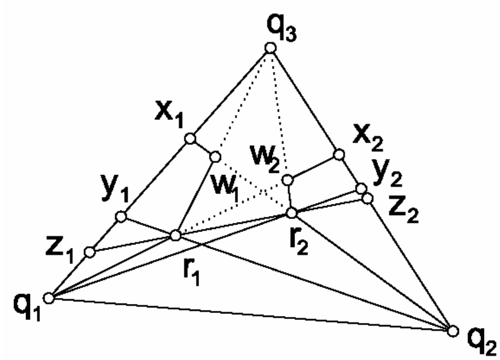

Figure 5

We show that if U has no sideline that separates U from an edge of Q, then S contains a hexagon. Indeed, if every sideline of U separates U from exactly one vertex of Q, then, by the pigeon-hole principle, Q has a vertex, say q_3, such that at least two sidelines of U separate U from it. This yields that there are two sidelines passing through consecutive edges of U that separate U from only q_3. Let these edges be $[r_{i-1}, r_i]$ and $[r_i, r_{i+1}]$. Then we have $[q_1, r_{i+1}, r_i, r_{i-1}, q_2] * [q_2, r_{i-1}, q_3] * [q_3, r_{i+1}, q_1]$. Hence, we may assume that U has a sideline that separates U from an edge of Q. Without loss of generality, let this sideline pass through the edge $[r_1, r_2]$ and let it separate U from $[q_1, q_2]$.

For every $3 \neq i \neq j \neq 3$, let x_i, y_i, and z_i denote the intersection point of the segment $[q_i, q_3]$ with the line $L(q_j, r_j)$, $L(q_j, r_i)$, and $L(r_1, r_2)$, respectively, and let w_i denote the intersection point of $[r_i, q_3]$ and $L(q_j, r_j)$ (cf. Figure 5). If some point $u \in R$ is beyond exactly the edge $[r_1, r_2]$ of $[q_1, q_2, r_2, r_1]$, then we have $[q_1, r_1, u, r_2, q_2] * [q_2, r_2, q_3] * [q_3, r_1, q_1]$. If $u \in R$ is beyond exactly the edge $[r_1, q_3]$ of $[q_1, r_1, q_3]$, then $[q_1, r_1, u, q_3] * [q_3, s, q_2] * [q_2, r_2, r_1, q_1]$ for $s = u$ or $s = r_2$. Hence, by symmetry, we may assume that r_3, r_4, and r are in one of the quadrangles $[r_i, w_i, x_i, z_i]$ for $i = 1$ or 2, or in $[q_1, q_2, z_2, z_1]$.

Assume that $r_3 \in [r_1, w_1, x_1, z_1]$. If $L^+(r_4, r_3) \cap [q_1, q_3] \neq \emptyset$, then $[q_3, r_3, r_4, r_1, r_2] * [r_2, r_1, q_1] * [q_1, r_3, q_3]$. If $L^-(r_4, r_3) \cap [q_1, q_3] \neq \emptyset$, then $[q_1, r_4, r_3, r_2, q_2] * [q_2, r_2, q_3] * [q_3, r_4, q_1]$. If $L(r_4, r_3) \cap [q_1, q_3] = \emptyset$, then $[q_1, r_1, r_2, q_2] * [q_2, r_2, q_3] * [q_3, r_3, r_4, q_1]$. Thus, we may assume that $r_3 \in [r_2, z_2, x_2, w_2]$. Since $r_3 \in [r_2, y_2, z_2]$ yields $[q_3, r_4, r_1, r_2, r_3] * [r_3, r_2, q_1] * [q_1, r_4, q_3]$, we may assume that $r_3 \in [r_2, w_2, x_2, y_2]$, and (by symmetry) that $r_4 \in [r_1, w_1, x_1, y_1]$.

Assume that $r \in [r_1, w_1, x_1, y_1]$. If $[r_1, r_2, r_4, r]$ is a quadrilateral, then we may apply an argument similar to that in the previous paragraph. Thus, we may assume that $r_4 \in [r_1, r_2, r]$. This yields $[r, r_4, r_1, q_1] * [q_1, r_1, r_2, q_2] * [q_2, r_3, q_3] * [q_3, r_3, r_2, r_4, r]$. Hence, $r \in [q_1, q_2, z_2, z_1]$.

If $r \in [q_1, r_1, z_1]$, then $[q_1, r, r_1, r_2, q_2] * [q_2, r_2, q_3] * [q_3, r, q_1]$. Let $r \in [q_1, q_2, r_2, r_1]$. If $L(q_3, r_4)$ does not separate q_1 and r, then $[q_1, r, q_2] * [q_2, r_1, r_4, q_3] * [q_3, r_4, r, q_1]$. Otherwise, we may suppose that $L^+(r, r_4) \cap [q_1, q_3] \neq \emptyset$. By symmetry, we also obtain that $L^+(r, r_3) \cap [q_2, q_3] \neq \emptyset$.

Assume that $r \in [q_1, r_1, r_2]$. Then, we observe that $U' = [r, r_2, r_3, r_1]$ is an empty quadrilateral, and $L(r, r_2)$ separates U' from $[q_1, q_2]$. Since

$R \cap [q_1, r, r_2, q_2] = \emptyset$, an argument applied for U', similar to that applied for U, yields a hexagon. Hence, $r \in [q_1, r_1, q_2] \cap [q_1, r_2, q_2]$. Then $L^+(r_3, r) \cap [q_1, q_2] \neq \emptyset \neq L^+(r_4, r) \cap [q_1, q_2]$. Now, we apply Lemma 3 with $V(P) \cup V(Q) \cup \{r_3, r_4, r\}$ as S. □

Definition 1. Let $A, B \subset \mathbb{E}^2$ be sets of points in general position. Suppose that there is a bijective function $f : A \to B$ such that, for any $a_1, a_2, a_3 \in A$, the ordered triples (a_1, a_2, a_3) and $(f(a_1), f(a_2), f(a_3))$ have the same or the opposite orientation, independently of the choice of a_1, a_2, and a_3. Then we say that A and B are *identical*.

We note that if A and B are identical, then $A' \subset A$ is a k-gon, if and only if, $f(A')$ is a k-gon.

Let $\widetilde{S}$ be a set of less than thirteen points such that $[\widetilde{S}]$ is a pentagon, $[\widetilde{S} \setminus V(\widetilde{S})]$ is a triangle, and $\widetilde{S}$ does not contain a hexagon. Using Lemma 4, we may characterize the possible configurations for $\widetilde{S} \setminus V(\widetilde{S})$. Lemma 5 summarizes our work. We sketch its proof.

Lemma 5. *Let $\widetilde{S} \subset \mathbb{E}^2$ be a set of fewer than thirteen points in general position such that $[\widetilde{S}]$ is a pentagon, $Q = [\widetilde{S} \setminus V(\widetilde{S})]$ is a triangle, and $\widetilde{S}$ does not contain a hexagon. Then Q is identical to one of the sets in Figure* 6.

Proof. Let the vertices of Q be q_1, q_2, and q_3 in counterclockwise cyclic order, and let $R = S \cap \operatorname{int} Q$. If $\operatorname{card} R \leq 2$, the assertion readily follows. Let us assume that card $R = 3$ and that the vertices of $[R]$ are r_1, r_2, and r_3 in counterclockwise cyclic order. By Lemma 3, we may assume that there is a sideline of R, that separates exactly two vertices of Q from R. Let this line be $L(r_1, r_2)$, and let it separate q_1 and q_2 from R. Note that if $r_3 \in [r_1, r_2, q_3]$, the assertion follows by an argument similar to that in the third paragraph of the proof of Lemma 4. Without loss of generality, we may assume that $L(q_3, r_1)$ separates r_3 from q_2. If $L(q_2, r_1)$ separates r_3 from q_3, then Q is a type 3b configuration. Otherwise, Q is a type 3a configuration.

The proof for the case card $R = 4$ is similar to the proof in the previous case, and hence we omit it. □

This list helps us to exclude some other cases from our investigation. If a set is identical to one of the sets in Figure 6, we say that its type is the type of the corresponding set in the figure.

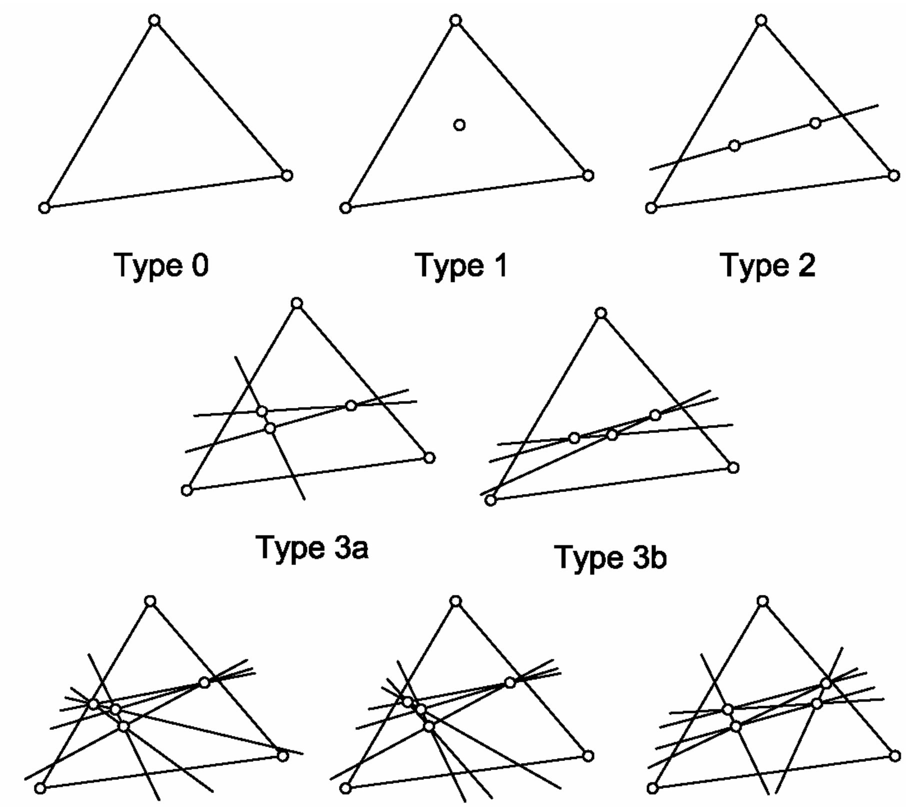

Figure 6

Lemma 6. *Let* $S \subset \mathbb{E}^2$ *be a set of seventeen points in general position such that* $P = [S]$ *is a pentagon and* $Q = [S \setminus V(P)]$ *is a quadrilateral. Then S contains a hexagon.*

Proof. Consider a diagonal D of Q. By Lemma 4, we may assume that D divides Q into two triangles that contain exactly four points of S in their interiors, and both these triangles have to be either type 4a, 4b, or 4c. Let us observe that if both triangles contain a pair of points such that the line passing through them does not intersect D, then these two pairs of points and the two endpoints of D are in convex position. Hence, we may assume that, in at least one of the triangles, each line passing through two points intersects D.

Since there is, in a type 4c set, no edge of the convex hull that meets all the lines that pass through two of its points, we may assume that the set of the points in one of the triangles is type 4a or 4b, and that D is the left edge of one of the triangles in Figure 6. We also observe that configurations of type 4a or 4b are almost identical, the only difference is that the line passing through the two points closest to the left edge of the triangle intersects the bottom or the right edge of the triangle. Thus, we may handle these two cases together, if we leave it open which edge this line intersects.

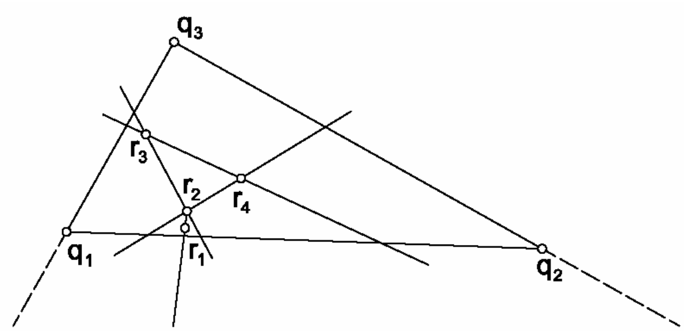

Figure 7

We denote our points as in Figure 7 with $D = [q_1, q_2]$, and let $L = L(r_1, r_2)$. Observe that L divides the set of points, beyond exactly the edge $[q_1, q_2]$ of $[q_1, q_2, q_3]$, into two connected components. If a point p is in the component that contains q_1, respectively q_2, in its boundary, then we say that p is on the *left-hand side*, respectively *right-hand side*, of L. Let $B = (Q \cap S) \setminus [q_1, q_2, q_3]$. Observe that, as $\operatorname{card}(Q \cap S) = 12$

and $\operatorname{card}(S \cap [q_1, q_2, q_3]) = 7$, we have that card $B = 5$ and every point of B is either on the left-hand side or on the right-hand side of L. By the pigeon-hole principle, there are three points of B that are on the same side of L. Let us denote these points by s_1, s_2, and s_3.

Assume that s_1, s_2, and s_3 are on the left-hand side of L. Observe that if $L(s_i, s_j)$ and $[q_1, r_1]$ are disjoint for some $i \neq j$, then $[q_1, s_i, s_j, r_1, r_2, r_3]$ is a hexagon. Thus, we may relabel s_1, s_2, and s_3 such that $s_3 \in [q_1, r_1, s_2] \subset [q_1, r_1, s_1]$. This yields that either $[s_1, s_2, s_3, q_1]$ or $[s_1, s_2, s_3, r_1]$ is a quadrilateral. If $[s_1, s_2, s_3, q_1]$ is a quadrilateral, then $[s_1, s_2, s_3, q_1] * [q_1, s_3, r_1, r_2, r_3] * [r_3, r_4, q_2] * [q_2, r_1, s_2, s_1]$. If $[s_1, s_2, s_3, r_1]$ is a quadrilateral, then $[s_1, s_2, s_3, r_1, q_2] * [q_2, r_4, r_3] * [r_3, r_2, r_1, s_3, q_1] * [q_1, s_2, s_1]$.

Let s_1, s_2, and s_3 be on the right-hand side of L. Observe that if $L(s_i, s_j)$ and $[q_2, r_1]$ are disjoint for some $i \neq j$, then $[q_2, s_i, s_j, r_1, r_2, r_4]$ is a hexagon. Hence, we may assume that $s_3 \in [q_2, r_1, s_2] \subset [q_2, r_1, s_1]$. Then $[s_1, s_2, s_3, q_2]$ or $[s_1, s_2, s_3, r_1]$ is a quadrangle. If $[s_1, s_2, s_3, q_2]$ is a quadrilateral, then $[s_1, s_2, s_3, q_2] * [q_2, s_3, r_1, r_2, r_4] * [r_4, r_2, q_1] * [q_1, r_1, s_2, s_1]$. If $[s_1, s_2, s_3, r_1]$ is a quadrilateral, then $[s_1, s_2, s_3, r_1, q_1] * [q_1, r_2, r_4] * [r_4, r_2, r_1, s_3, q_2] * [q_2, s_2, s_1]$. $\square$

Lemma 7. *Let* $S \subset \mathbb{E}^2$ *be a set of points in general position such that* $P = [S]$ *and* $Q = [S \setminus V(P)]$ *are pentagons, and* $S \setminus (V(P) \cup V(Q))$ *has a subset of type* 3a, *or a subset identical to the point set in Figures* 9, 10, *or* 11. *Then S contains a hexagon.*

Proof. Let R denote the subset of $S \setminus (V(P) \cup V(Q))$ that is either of type 3a, or is identical to the point set in Figures 9, 10 or 11. Let q_1, q_2, q_3, q_4, and q_5 denote the vertices of Q in counterclockwise cyclic order.

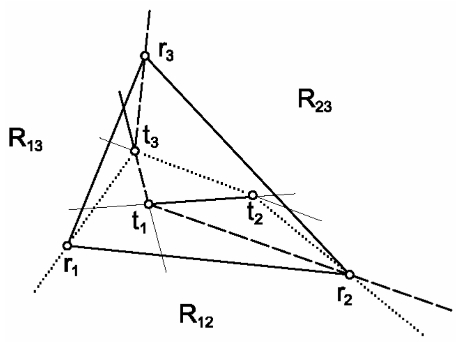

Figure 8. A type 3a set with the notation of Lemma 7.

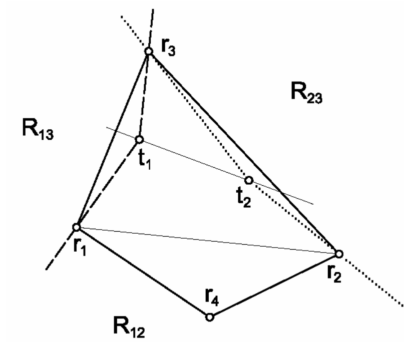

Figure 9

Assume that R is of type 3a. Let us denote the points of R as in Figure 8. Let R_{12}, R_{23}, and R_{13} denote, respectively, the set of points that are beyond exactly the edge $[r_1, r_2]$ of $[r_2, t_2, t_3, r_1]$, the edge $[r_2, r_3]$ of $[r_2, t_1, t_3, r_3]$, and the edge $[r_1, r_3]$ of $[r_1, t_3, r_3]$. If $\text{card}(R_{12} \cap V(Q)) \geq 2$, $\text{card}(R_{23} \cap V(Q)) \geq 2$, or $\text{card}(R_{13} \cap V(Q)) \geq 3$, then S contains a convex hexagon. Otherwise, there is a vertex q_i of Q in the convex domain bounded by the half-lines $L^-(r_2, t_1)$ and $L^-(r_2, t_2)$, from which we obtain $[r_1, t_1, r_2, q_i] * [q_i, r_2, t_2, r_3] * [r_3, t_3, r_1]$.

Let us assume that R is the set in Figure 9 and denote the points of R as indicated. Let R_{12}, R_{23}, and R_{13} denote, respectively, the set of points that are beyond exactly the edge $[r_1, r_2]$ of $[r_1, t_1, t_2, r_2]$, the edge $[r_2, r_3]$ of $[r_2, t_2, r_3]$, and the edge $[r_1, r_3]$ of $[r_1, t_1, r_3]$. If $\text{card}(R_{12} \cap V(Q)) \geq 2$, $\text{card}(R_{23} \cap V(Q)) \geq 3$, or $\text{card}(R_{13} \cap V(Q)) \geq 3$, then S contains a hexagon. Hence, we may assume that $q_1 \in R_{12}$, $\{q_2, q_3\} \subset R_{23}$, $\{q_4, q_5\} \subset R_{13}$, and there is no vertex of Q in $R_{23} \cap R_{13}$. If $L(q_1, r_4)$ does not intersect the interior of $[R]$, then the convex hull of $[t_1, t_2, r_2, r_1]$ and $[r_4, q_1]$ is a hexagon.

Let $r_4 \in [q_1, r_1, r_2]$. If $L(r_4, r_1)$ does not separate q_5 and q_1, and $L(r_4, r_2)$ does not separate q_2 and q_1, then $[q_1, r_4, r_2, q_2] * [q_2, r_2, t_2, r_3]$

$*[r_3, t_1, r_1, q_5]*[q_5, r_1, r_4, q_1]$. Thus, we may assume that, say, $L(r_4, r_1)$ separates q_5 and q_1. If $L(r_2, r_3)$ separates q_4 and R, then $[q_4, r_3, r_2]*[r_2, t_2, t_1, r_1]*[r_1, t_1, r_3, q_4]$. If $L(r_2, r_3)$ does not separate q_4 and R, then $[r_4, r_2, r_3, q_4, q_5, r_1]$ is a hexagon.

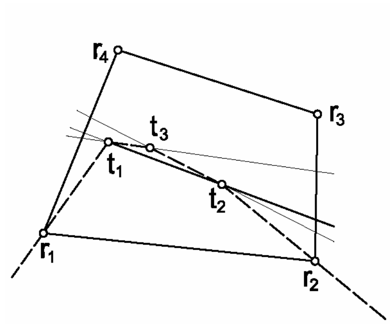

Figure 10

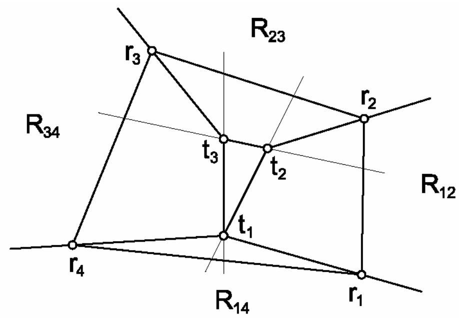

Figure 11

Assume that R is the set in Figure 10 and denote the points of R as indicated. We may clearly assume that there is no vertex of Q beyond exactly the edge $[r_1, r_2]$ of $[r_1, r_2, t_2, t_3, t_1]$. Hence, there is an edge, say $[q_1, q_2]$, that intersects both rays $L^-(r_1, t_1)$ and $L^-(r_2, t_2)$. If $L(r_1, r_2)$ separates R from both q_1 and q_2, then $[q_1, r_1, r_2, q_2]*[q_2, r_2, t_2, r_3]*[r_3, t_2, t_1, r_4]*[r_4, t_1, r_1, q_1]$. Hence, we may assume that $L(r_1, r_2)$ does not separate R, say, from q_2. If $L(t_2, t_3)$ does not separate r_2 and q_2, then $[r_1, r_2, q_2, t_2, t_3, t_1]$ is a hexagon. If $L(t_2, t_3)$ separates r_2 and q_2, then $[q_1, r_2, q_2]*[q_2, t_2, t_3, r_4]*[r_4, t_1, r_1, q_1]$.

We are left with the case when R is the set in Figure 11 with points as indicated. Let R_{12}, R_{23}, R_{34}, and R_{14} denote, respectively, the set of points that are beyond exactly the edge $[r_1, r_2]$ of $[r_1, r_2, t_2, t_1]$, the edge $[r_2, r_3]$ of $[r_2, t_2, t_3, r_3]$, the edge $[r_3, r_4]$ of $[r_3, t_3, t_1, r_4]$, and the edge $[r_1, r_4]$ of $[r_4, t_1, r_1]$. If $\mathrm{card}(R_{i(i+1)} \cap V(Q)) \geq 2$ for some $i \in \{1, 2, 3\}$, then S contains a hexagon. Otherwise, R_{14} contains at least two vertices of Q, which we denote by q_1 and q_2. If both q_1 and q_2 are beyond exactly the edge $[r_1, r_4]$ of $[r_1, t_2, t_3, r_4]$, then $[t_2, t_3, r_4, q_1, q_2, r_1]$ is a hexagon. Thus, we may assume that, say, q_1 is beyond exactly the edge $[r_3, r_4]$ of $[r_3, t_3, r_4]$. From this, it follows that $[r_3, t_3, r_4, q_1] * [q_1, r_4, t_1, r_1] * [r_1, t_1, t_2, r_2] * [r_2, t_2, t_3, r_3]$. □

Proof of Theorem 1. Let $Q = [S \setminus V(P)]$, $R = [S \setminus (V(P) \cup V(Q))]$, and $T = S \setminus (V(P) \cup V(Q) \cup V(R))$. If Q is a triangle, then we apply Lemma 4. If Q is a quadrilateral, then we apply Lemma 6. Let Q be a pentagon. If R is a triangle, then, by Lemma 5, $V(R) \cup T$ has types 4a, 4b, or 4c, and thus, it contains a type 3a subset, and the assertion follows from Lemma 7. If R is a quadrilateral, then $V(R) \cup T$ contains a subset identical to the set in Figures 9, 10, or 11, and we may apply Lemma 7.

Let R be a pentagon. We note that T contains two points, say, t_1 and t_2.

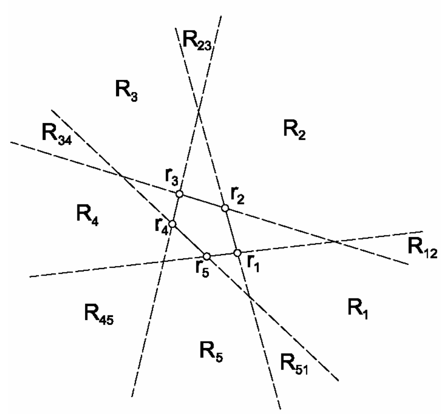

Figure 12

Let q_1, q_2, q_3, q_4, q_5, and r_1, r_2, r_3, r_4, r_5 denote, respectively, the vertices of Q and R in counterclockwise cyclic order. If some q_i is beyond exactly one edge of R, then $[R, q_i]$ is a hexagon. Thus, we may assume that every vertex of Q is beyond at least two edges of R. Observe that there is no point on the plane that is beyond all five edges of R. If some q_i is beyond all edges of R but one, say $[r_1, r_5]$, then we obtain $[r_1, r_2, r_3, r_4, r_5] * [r_5, r_4, q_i] * [q_i, r_2, r_1]$. Hence, we may assume that every vertex of Q is beneath at least two edges of R.

For $1 \leq i \leq 5$, let R_i denote the set of points that are beyond the two edges of R that contain r_i and beneath the other three edges of R, and let $R_{i(i+1)}$ denote the set of points that are beyond the edges of R that contain r_i or r_{i+1}, and beneath the other two edges of R (cf. Figure 12). We call $R_{(i-1)i}$ and $R_{i(i+1)}$ *consecutive regions*.

Assume that two distinct and nonconsecutive regions contain vertices of Q, say, $q_k \in R_{51}$ and $q_l \in R_{23}$. Since every vertex of Q is beneath at least two edges of R, q_k and q_l are distinct points. If there is a vertex q_h of Q in $R_{34} \cup R_4 \cup R_{45}$, then $[q_l, r_3, r_4, q_h] * [q_h, r_4, r_5, q_k] * [q_k, r_1, r_2, q_l]$. Let $V(Q) \cap (R_{34} \cup R_4 \cup R_{45}) = \emptyset$. Then exactly one edge of Q intersects $R_{34} \cup R_4 \cup R_{45}$. Let us denote this edge by $[q_m, q_{m+1}]$. If $q_m \in R_{23}$, then $[q_{m+1}, r_4, q_m] * [q_m, r_2, r_1] * [r_1, r_2, r_3, r_4, q_{m+1}]$. Let $q_m \in R_3$ and, by symmetry, $q_{m+1} \in R_5$. If there are at least three vertices of Q in $R_2 \cup R_{23} \cup R_3$ or in $R_1 \cup R_{15} \cup R_5$, then $V(Q) \cup V(R)$ contains a hexagon. Hence, we may assume that a vertex q_g of Q is in R_{12}. Since every vertex of Q is beneath at least two edges of Q, the sum of the angles of R at r_1 and r_2 is greater than π, which implies that $L(r_1, r_2)$ separates R and q_g. Thus, we have $[q_g, r_2, r_3, q_m] * [q_m, r_4, q_{m+1}] * [q_{m+1}, r_5, r_1, q_g]$.

Assume that two consecutive regions contain vertices of Q, say, $q_k \in R_{51}$ and $q_l \in R_{12}$. If $V(Q) \cap (R_{23} \cup R_{34} \cup R_{45}) \neq \emptyset$, then we may apply the argument in the previous paragraph. Let $V(Q) \cap (R_{23} \cup R_{34} \cup R_{45}) = \emptyset$. If at least four vertices of Q are beneath the edge $[r_3, r_4]$ of R, then these vertices, together with r_3 and r_4, are six points in convex position. Hence, we may assume that $R_3 \cup R_4$ contains at least two vertices of Q. Let us denote these vertices by q_e and q_f. If $q_e, q_f \in R_3$, then $[r_1, r_2, q_e, q_f, r_4, r_5]$ is a hexagon. Thus, we may clearly assume that, say, $q_e \in R_3$ and $q_f \in R_4$. Then we have $[q_l, r_2, r_3, q_e] * [q_e, r_3, r_4, q_f] * [q_f, r_4, r_5, q_k] * [q_k, r_5, r_1, q_l]$.

Assume that $R_{i(i+1)}$ contains a vertex of Q for some i, say, $q_1 \in R_{51}$. By the preceeding, no vertex of Q is in $R_{12} \cup R_{23} \cup R_{34} \cup R_{45}$. An argument similar to that used in the previous paragraph yields the existence of a hexagon if R_2, R_3, or R_4 contains no vertex of Q. Let $q_k \in R_2$, $q_l \in R_3$, and $q_m \in R_4$. Then $[q_l, r_1, r_2, q_k] * [q_k, r_2, r_3, q_l] * [q_l, r_3, r_4, q_m] * [q_m, r_4, r_5, q_1]$.

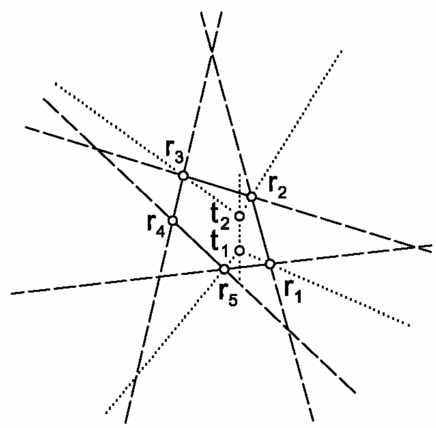

Figure 13

We have now arrived at the case that each vertex of Q is beyond exactly two edges of R. Clearly, we may assume that $q_i \in R_i$ for each i. If $L(t_1, t_2)$ intersects two consecutive edges of R, then S contains a hexagon. Hence, we may assume that, say, $L^+(t_1, t_2) \cap [r_2, r_3] \neq \emptyset$, and $L^-(t_1, t_2) \cap [r_5, r_1] \neq \emptyset$ (cf. Figure 13). If both q_1 and q_2 are beyond exactly the edge $[r_1, r_2]$ of $[r_1, t_1, t_2, r_2]$, then we have a hexagon. If neither point is beyond exactly that edge, then $[q_1, r_1, r_2, q_2] * [q_2, r_2, t_2, r_3] * [r_3, t_2, t_1, r_5] * [r_5, t_1, r_1, q_1]$. Thus, we may assume that q_1 is beyond exactly the edge $[r_1, r_2]$ and q_2 is not. If q_5 is beyond exactly the edge $[r_4, r_5]$ of $[r_4, r_5, t_1, t_2, r_3]$, then $[q_5, r_5, t_1, t_2, r_3, r_4]$ is a hexagon. Hence, we may assume that q_5 is beyond exactly the edge $[r_1, r_5]$ of $[r_1, t_1, r_5]$ and, similarly, that q_3 is beyond exactly the edge $[r_2, r_3]$ of $[r_2, t_2, r_3]$. From this, we obtain that $[q_3, r_3, r_4, q_4] * [q_4, r_4, r_5, q_5] * [q_5, r_5, r_1, q_1] * [q_1, r_1, r_2, q_2] * [q_2, r_2, t_2, r_3, q_3]$. □

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